Riemann Curvature in Collective Spaces

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Abstract. For a generally curved collective space an additional interaction with the metric is investigated, by introducing the collective curvature scalar. The coupling strength is determined by a fit on AME2003 ground state masses. An extended finite-range droplet model including curvature is presented. Significant improvements for light nuclei and nuclei in the trans-fermium region are interpreted as an evidence for the existence of this new interaction.

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1. Introduction

The use of collective models for a description of collective aspects of nuclear motion has proven considerably successful during the past decades. Calculating life-times of heavy nuclei[1], fission yields[2] or giving insight into phenomena like cluster-radioactivity[3] or bimodal fission, remarkable results have been achieved by introducing an appropriate set of collective coordinates, for instance length, deformation, neck or mass-asymmetry[4] for a given nuclear shape and investigating its dynamic properties.

Introducing a collective potential V_0, depending on N collective coordinates $\{q^i, i = 1, ..., N\}$,

$$V_0(q^i) = E_{macro}(q^i) + E_{shell}(q^i) + E_{pairing}(q^i) \tag{1}$$

equations of motion are derived by quantizing[5] the classical kinetic energy T

$$T = \frac{1}{2} B_{ij}\, \dot{q}^i \dot{q}^j \tag{2}$$

which yields a collective Schrödinger-equation.

There are several common methods to generate the collective mass parameters B_{ij}, e.g. on a microscopic level the cranking model[6] or irrotational flow models are used. We want to emphasize the fact, that via the relation (m_A is the mass of the nucleus, $m_u = 931.5$ MeV is the mass unit and A is the number of nucleons)

$$B_{ij} = m_A\, g_{ij} = m_u\, A\, g_{ij} \tag{3}$$

the collective masses may be interpreted geometrically, defining the metric tensor g_{ij}, which fully determines the geometric properties of collective Riemann space.

The resulting collective Schrödinger-equation

$$\hat{S}_0 \Psi(q^i, t) = \left(-\frac{\hbar^2}{2\, m_A} \frac{1}{\sqrt{g}} \partial_i g^{ij} \sqrt{g} \partial_j - i\hbar\partial_t + V_0\right) \Psi(q^i, t) = 0 \tag{4}$$

is the central ingredience for a discussion of collective phenomena.

A different approach, which results in the same equations of motions, starts with the Lagrange density $\mathcal{L}_0$

$$\mathcal{L}_0 = \frac{\hbar^2}{2\, m_A} g^{ij} (\partial_i \Psi^*)(\partial_j \Psi) + \frac{i\hbar}{2}(\Psi^* \frac{\partial \Psi}{\partial t} - \frac{\partial \Psi^*}{\partial t}\Psi) - \Psi^* V_0 \Psi \tag{5}$$

Variation with respect to Ψ^* and Ψ yields the above Schrödinger-equation.

From this point of view, it is remarkable, that until now obvious extensions of this Lagrange density have been discussed in other branches of physics, e.g. cosmology or string theory, but, until now, have been neglected for collective models.

2. Curvature in collective space

In order to investigate the influence of curvature in collective space, we consider an additional interaction with the collective metric, which is determined by the collective mass parameters.

We extend the Lagrangian density

$$\mathcal{L} = \mathcal{L}_0 - \frac{\hbar^2}{2\, m_A} \xi \Psi^* R \Psi \tag{6}$$

introducing the Einstein curvature scalar R as an invariant measure for collective curvature. The coupling strength is parametrized with ξ.

Variation of this Lagrangian density results in an additional potential term

$$V = V_0 + \frac{\hbar^2}{2 m_A} \xi R \tag{7}$$

Since the collective mass parameters are known, R can be calculated, using the Christoffel symbols of second kind [7]

$$\Gamma^{\mu}_{\kappa\sigma} = \frac{1}{2} g^{\nu\mu} (\partial_\kappa g_{\nu\sigma} + \partial_\sigma g_{\nu\kappa} - \partial_\nu g_{\kappa\sigma}) \tag{8}$$

As a starting point the Riemann curvature tensor is given by

$$R^{\alpha}{}_{\eta\beta\gamma} = \partial_\gamma \Gamma^{\alpha}_{\beta\eta} - \partial_\beta \Gamma^{\alpha}_{\eta\gamma} + \Gamma^{\alpha}_{\tau\gamma}\Gamma^{\tau}_{\beta\eta} - \Gamma^{\alpha}_{\tau\beta}\Gamma^{\tau}_{\gamma\eta} \tag{9}$$

which may be contracted to get the Ricci tensor

$$R_{\eta\gamma} = R^{\alpha}{}_{\eta\alpha\gamma} \tag{10}$$

and finally

$$R = R^{\eta}{}_{\eta} \tag{11}$$

After introducing a curvature term formally, its explicit form depends on a specific choice of collective coordinates.

In order to examine the consequences and physical interpretation of this additional new term we choose the symmetric two-center shell model including elliptical deformations[8], which, for reasons of clarity, will be solved analytically. It has widely been used in the description of symmetric fusion reactions and contains the Nilsson model as a limiting case, which will turn out to be a useful property for a physical interpretation.

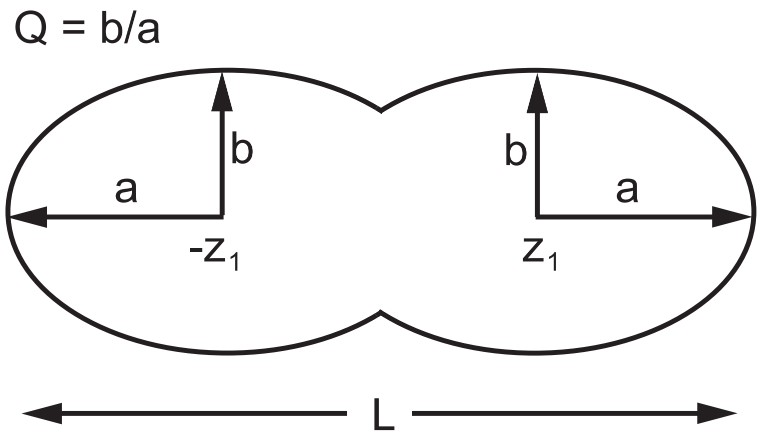

Figure 1. geometry of the symmetric two-center shell model

3. Exact solution for the symmetric two-center shell model

As an illustrative, exactly solvable scenario we consider the nuclear shape given by two intersecting rotationally symmetric ellipsoids. Introducing two collective coordinates q^i namely, the ellipsoidal deformation $Q = b/a$ and the total elongation L the shape is given by:

$$P(z, q^i) = Q\sqrt{a^2 - (z \mp z_1)^2} \tag{12}$$

The geometric quantities semi axis a and center position of ellipsoids z_1 are determined by the definition of L and by forcing volume conservation, which is defined e.g. for connected fragments

$$L = 2(a + z_1) \tag{13}$$

$$V = (4/3)\pi {R_0}^3 = 2\,(2/3)\,\pi Q^2(a^3 + \frac{3}{2}a^2 z_1 - \frac{1}{2}z_1^3) \tag{14}$$

These equations may be simplified introducing the dimensionless quantities

$$\alpha \;= Q^{2/3}\frac{a}{R_0} \tag{15}$$

$$\gamma_1 = Q^{2/3}\frac{z_1}{R_0} \tag{16}$$

$$\lambda \;= Q^{2/3}\frac{L}{2R_0} \tag{17}$$

Equation(17) defines a transformation to a new set of coordinates(λ, Q), where the range from spherical compound nucleus $\lambda = 1$ up to scission point configuration $\lambda = 4^{1/3}$ is independent from Q. Results depend on λ only

$$\alpha \;= \frac{2 + \lambda^3}{3\,\lambda^2} \tag{18}$$

$$\gamma_1 = \frac{2\,(\lambda^3 - 1)}{3\,\lambda^2} \tag{19}$$

Thus, the shape geometry is fully determined for a given set of collective coordinates.

We apply the Werner-Wheeler-formalism[9] for the intermediate set of collective coordinates (λ, Q) to calculate the collective masses M_{ij}. We choose this method, since masses are determined by shape geometry only and the procedure itself is well defined. Using the abbreviation

$$ln = \ln(\frac{4-\lambda^3}{4+2\,\lambda^3}) \tag{20}$$

according to equation(3) the metric tensor g_{ij} results as

$$\begin{aligned} g_{\lambda\lambda} = & \frac{{R_0}^2 (2+\lambda^3)^2}{324\, Q^{4/3}\, \lambda^{12}} \\ & \times\{3\,(Q^2-16)\,\lambda^9 + 4\,(Q^2-4)\,[24\,\lambda^3 - 3\,\lambda^6 + (\lambda^3-4)^2\,(2+\lambda^3)\,ln]\} \end{aligned} \tag{21}$$

$$g_{\lambda Q} = -\frac{{R_0}^2}{108\, Q^{7/3}\, \lambda^2} (2+\lambda^3)\,[6\,\lambda^3 + Q^2\,(4-\lambda^3)] \tag{22}$$

$$g_{QQ} = \frac{{R_0}^2}{810\, Q^{10/3}\, \lambda} [12\,\lambda^3\,(5+\lambda^3) + Q^2\,(40 - 5\,\lambda^3 + \lambda^6)] \tag{23}$$

Performing a coordinate transformation from the coordinate set (λ, Q) to the original (L, Q) using

$$g_{ij} = \frac{\partial x^m}{\partial x^i}\frac{\partial x^n}{\partial x^j} g_{mn} \tag{24}$$

yields the final result for connected shapes in the range $1 \le \lambda \le 4^{1/3}$

$$\begin{aligned} g_{LL} = & \frac{(2+\lambda^3)^2}{1296\lambda^{12}} \\ & \times\{3(Q^2-16)\lambda^9 + 4(Q^2-4)[8\lambda^3(3-\lambda^3) + (4-\lambda^3)^2(2+\lambda^3)ln]\} \end{aligned} \tag{25}$$

$$\begin{aligned} g_{LQ} = & \frac{R_0\,(2+\lambda^3)}{1944\, Q^{5/3}\lambda^{11}} \\ & \times\{3\lambda^3[(Q^2+14)\lambda^9 + 8(Q^2-4)(16+6\,\lambda^3-3\lambda^6)] + 8(Q^2-4)(-8-2\lambda^3+\lambda^6)^2 ln\} \end{aligned} \tag{26}$$

$$\begin{aligned} g_{QQ} = & \frac{{R_0}^2}{7290\, Q^{10/3}\lambda^{10}} \\ & \times\{(48+69\,Q^2)\lambda^{15} + (Q^2-4)[15\lambda^3(256+224\lambda^3-31\lambda^9) + 40(\lambda^3-4)^2(2+\lambda^3)^3 ln]\} \end{aligned} \tag{27}$$

A similar calculation for separated fragments with $\lambda \ge 4^{1/3}$ yields

$$g_{LL} = \frac{1}{4} \tag{28}$$

$$g_{LQ} = \frac{R_0/3}{2^{1/3}\, Q^{5/3}} \tag{29}$$

$$g_{QQ} = \frac{2^{1/3}\,(12+Q^2){R_0}^2}{45\, Q^{10/3}} \tag{30}$$

Given the metric tensor g_{mn}, the curvature scalar $R(L,Q)$ can easily be calculated. Using the abbreviations

$$\begin{aligned} k_0 = & 1638400 + 81920\,\lambda^3 - 654336\,\lambda^6 + 96256\,\lambda^9 + 70592\,\lambda^{12} - 9216\,\lambda^{15} \\ & -3119\,\lambda^{18} + 578\,\lambda^{21} \end{aligned} \tag{31}$$

$$k_1 = 8\,(-1+\lambda)\,(2+\lambda)\,[4+(-2+\lambda)\,\lambda]\,(1+\lambda+\lambda^2)\,(4-\lambda^3) \tag{32}$$

$$k_2 = 10240 + 5120\,\lambda^3 - 2400\,\lambda^6 - 760\,\lambda^9 + 13\,\lambda^{12} \tag{33}$$

$$k_3 = 32\,(-4+\lambda^3)^4\,(2+\lambda^3)^3\,[Q^2\,(-10+\lambda^3)^2 + 20\,(-8+7\,\lambda^3+\lambda^6)] \tag{34}$$

$$\begin{aligned} k_4 = & -163840\,(-8+5\,Q^2) - 20480\,(16+17\,Q^2)\,\lambda^3 \\ & +3072\,(-380+89\,Q^2)\,\lambda^6 + 640\,(-16+91\,Q^2)\,\lambda^9 - 32\,(-6086+1259\,Q^2)\,\lambda^{12} \\ & -24\,(-274+97\,Q^2)\,\lambda^{15} + (-6964+1477\,Q^2)\,\lambda^{18} - 4\,(-49+34\,Q^2)\,\lambda^{21} \end{aligned} \tag{35}$$

$$p_0 = (-4+\lambda^3)\,(640 - 54\,\lambda^6 + 17\,\lambda^9) \tag{36}$$

$$p_1 = -2560 + 1600\,\lambda^3 + 288\,\lambda^6 - 266\,\lambda^9 + 47\,\lambda^{12} \tag{37}$$

$$p_2 = 1280 + 96\,\lambda^3 - 192\,\lambda^6 + 13\,\lambda^9 \tag{38}$$

$$p_3 = (Q^2-4)\,(-4+\lambda^3)^2\,(2+\lambda^3)\,[12\,\lambda^3\,(5+\lambda^3) + Q^2\,(40-5\,\lambda^3+\lambda^6)] \tag{39}$$

$$t_1 = 9\,(k_1\,k_2 + k_0\,Q^2)\,\lambda^6 + 4\,ln\,[k_3\,ln + 3\,k_4\,\lambda^3\,(\lambda^3-4)] \tag{40}$$

$$t_2 = [-8\,ln\,p_3 + 3\,\lambda^3\,(-4\,p_1\,Q^2 + p_0\,Q^4 + 12\,p_2\,\lambda^3)]^2 \tag{41}$$

the curvature tensor for connected fragments is given by

$$R(L,Q) = -\frac{2160}{{R_0}^2} Q^{10/3} \frac{\lambda}{2+\lambda^3}\,\frac{t_1}{t_2} \tag{42}$$

and for separated fragments ($\lambda \geq 4^{1/3}$)

$$R(L,Q) = 0 \tag{43}$$

Thus, for connected fragments with $\lambda \leq 4^{1/3}$ a nonvanishing curvature scalar $R(L,Q)$ exists. This is, as $R \sim {R_0}^{-2}$ dependence indicates, a direct consequence of the volume conservation condition (see equation(14)). Figure(2) shows the elements of the metric tensor g_{ij} and the resulting curvature scalar R. For prolate and moderately oblate shapes ($Q \leq 1.7$) with a fixed Q, the curvature scalar starts with a negative value, which tends to 0 with increasing λ up to the scission point. For oblate shapes ($Q \geq 1.7$) with a fixed Q, R decreases with increasing λ down to the scission point.

The discontinuity of the curvature scalar at the scission point is a direct consequence of the underlying simple geometry, the derivative of the shape is not defined at the contact point of the two ellipsoids. This can be avoided by smoothing the shape appropriately, resulting in a smooth curvature term at the scission point, but the resulting model has to be solved numerically.

The curvature minimizing shape is not a sphere, but a slightly deformed, oblate shape, due to the fact, that the subject of our considerations is not the curvature of a given shape, but curvature of the collective space, generated by the Werner-Wheeler-masses.

In case of a single deformed ellipsoid ($\lambda = 1$) R reduces to:

$$R(Q) = \frac{-720\,Q^{16/3}\,(-25+36\,\log(2))\,(-67+96\,\log(2))}{{R_0}^2\,(2+Q^2)^2\,[-2+(Q^2-4)\,(-67+96\,\log(2))]^2} \tag{44}$$

and finally, for a sphere

$$R_{sphere} = -4.3599\frac{1}{{R_0}^2} \tag{45}$$

Since R is an invariant (transforms as a scalar) under coordinate transformations, the shape may be described by any appropriate set of coordinates, which obey the

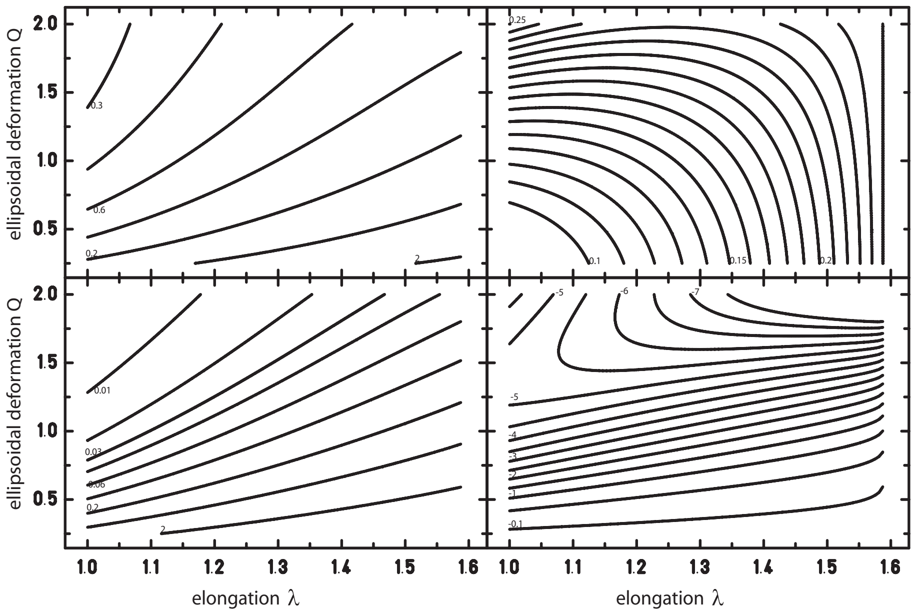

Figure 2. metric tensor components g_{LQ} (upper left), g_{LL} (upper right), g_{QQ} (lower left) and curvature scalar R (lower right) for the symmetric two center shell model, for $R_0 = 1$

transformation rule given in equation(24). Consequently, for the symmetric two center shell model, which we discussed here as an example, the coordinate sets (L, Q),(λ, Q), $(\lambda, \beta = 1/Q)$ or $(\Delta z, \beta)$ where Δz is the two-center distance, are equivalent. They lead to different mass parameters, but yield the same R. In that sense, the curvature scalar is a unique, outstanding property of a given shape geometry.

4. Determination of the coupling constant

In order to get an estimate for the curvature coupling constant, we will now investigate the influence of an additional curvature term by a fit of experimentally known ground state masses of nuclei.

For reasons of simplicity, we assume the ground state of nuclei being of ellisoidal form only, neglecting higher order multipoles. Therefore the shapes are described by $(\lambda = 1, Q)$, depending on only one collective coordinate Q.

We define the relative curvature energy B_R with respect to the spherical compound nucleus $(\lambda = 1, Q = 1)$ as

$$B_R = R(\lambda = 1, Q)/R_{sphere} \tag{46}$$

$$= 9\, Q^{16/3} \left(\frac{199 - 288 \log(2)}{(2 + Q^2)[266 - 67Q^2 + 96(Q^2 - 4) \log(2)]}\right)^2 \tag{47}$$

An additional curvature potential term $V_R(Q)$, using $R_0 = r_0 A^{1/3}$, is defined

$$V_R(Q) = +\frac{\hbar^2}{2m_A} \xi R_{sphere} B_R \tag{48}$$

Table 1. determination of volume-energy a_v, volume-asymmetry k_v, surface-energy a_s, surface-asymmetry k_s and curvature energy a_R constants within the original FRLDM, FRLDM2003 fitted with AME2003 experimental masses and FRLDMC, which corresponds to FRLDM plus curvature term and resulting root-mean-square deviations (rms) from AME2003 experimental data

constants	FRLDM	FRLDM2003	FRLDMC
a_v	16.00126 MeV	16.00496 MeV	16.01890 MeV
k_v	1.92240 MeV	1.93167 MeV	1.92882 MeV
a_s	21.18466 MeV	21.18770 MeV	21.25974 MeV
k_s	2.345 MeV	2.35968 MeV	2.34955 MeV
a_R	-	-	529.95850 MeV
rms	0.821 MeV	0.815 MeV	0.764 MeV

$$= -a_R B_R A^{-5/3} \tag{49}$$

Where we have introduced the curvature-energy constant a_R, which will be determined now.
Our choice for an appropriate macroscopic model is the finite range liquid drop model FRLDM. It is widely used and documented in detailref[10].
We will vary only a subset of parameters, namely, the volume-energy a_v, the volume-asymmetry k_v, the surface-energy a_s and the surface-asymmetry k_s constants, which generate the major contributions for the calculated masses, keeping all other parameters at their original values.
We define the finite range liquid drop model with curvature ($FRLDMC$):

$$FRLDMC(a_v, k_v, a_s, k_s, a_R, Q) = FRLDM(a_v, k_v, a_s, k_s, Q) + V_R(a_R, Q) \tag{50}$$

Theoretical masses m_{th} are then obtained, including unchanged shell corrections E^{FL}_{mic}

$$m_{th} = FRLDMC + E^{FL}_{mic} \tag{51}$$

Theoretical masses are compared with experimental masses and ground state deformations for 2091 nuclei in the range $8 \leq Z \leq 109$ from [11].

For conversion from quadrupole moments β_2 to ellipsoidal deformations Q we use the relation

$$Q = 1 + \frac{3}{2}\sqrt{\frac{5}{4\pi}}\beta_2 \tag{52}$$

As a measure for the quality of the fit, we tabulate the root mean square deviation

$$rms = \sqrt{\frac{1}{N}\sum^{N}(m_{exp} - m_{th})^2} \tag{53}$$

Results are listed in table 1. The first column tabulates the original FRLDM parameterset, followed by results for FRLDM2003, which corresponds to an actualized FRLDM-parameter set for AME2003 masses and finally results for FRLDMC, which corrsponds to the original FRLDM including the curvature term are presented.

The corresponding rms-values indicate a significant improvement of the new, extended FRLDMC-model. Especially for light nuclei and in the region of trans lead elements results are impressing, as shown in table 3, where errors for different Z-regions are listed.

Table 2. determination of volume-energy a_1, surface-energy a_2, charge-asymmetry c_a and curvature energy a_R constants within the original FRDM, FRDM2003 fitted with AME2003 experimental masses and FRDMC, which corresponds to FRDM plus curvature term and resulting root-mean-square deviations (rms) from AME2003 experimental data

constants	FRDM	FRDM2003	FRDMC
a_1	16.247 MeV	16.2401 MeV	16.2467 MeV
a_2	22.92 MeV	22.8812 MeV	22.9159 MeV
c_a	0.436 MeV	0.4368 MeV	0.4332 MeV
a_R	-	-	172.676 MeV
rms	0.679 MeV	0.674 MeV	0.655 MeV

For light nuclei this is due to the $A^{-5/3}$ behaviour of the curvature energy, since this term contributes most to the total binding energy for light nuclei, e.g. for ^{16}O = 5.21 MeV, while for heavy nuclei, this term becomes neglible e.g. for ^{208}Pb = 0.07 MeV. The improvement in the trans-lead region indicates, that a Riemann curvature term is a useful extension for a collective model.

Since overestimating masses for heavy nuclei is a known shortcoming of FRLDM, Nix et al.[10] introduced the finite range droplet model (FRDM), whose major improvement is an additional empirical exponential term of the form

$$-CA \exp^{-\gamma A^{1/3}} \bar{\epsilon} \tag{54}$$

Using original parameters, this term simulates an A-dependence, which is close to the collective curvature term, derived in this work.

Therefore we expect a reduced influence of an additional curvature term within the framework of FRDM. To proof this hypothesis, we define the finite range drop model with curvature ($FRDMC$), and vary with respect to the subset of most important parameters, a_1 volume-energy, a_2 surface-energy and c_a charge-asymmetry constants, keeping all other parameters fixed at their original values.

$$FRDMC(a_1, a_2, c_a, a_R, Q) = FRDM(a_1, a_2, c_a, Q) + V_R(a_R, Q) \tag{55}$$

Theoretical masses m_{th} are calculated, including unchanged shell corrections E_{mic}

$$m_{th} = FRLDMC + E_{mic} \tag{56}$$

For this scenario, the fit procedure was performed. Results are listed in table 2.

As exspected, the curvature-energy constant a_R is reduced by a factor 3, which results in an absolute contribution to the total binding energy of about 1.7 MeV for ^{16}O.

For light and trans-fermium nuclei we achieve is a significant improvement with the extended FRDMC, compared to the original $FRDM$. The additional curvature term makes the FRDMC the best model available for the description of ground state masses in the full range of the nuclear table.

Thus, within both extended models, the FRLDMC and the FRDMC, the existence of a curvature term is supported. The coupling strength ξ, derived from fits, setting $r_0 = 1.16$ fm and $m_u = 931.5$ MeV, results as $\xi = 7.8$ from $FRLDMC$ and $\xi = 2.5$ respectively, if $FRDMC$ is used as underlying model.

Table 3. root-mean-square deviations (rms) from AME2003 experimental data in MeV for different Z-Regions

Z/model	8-20	20-40	40-60	60-80	80-100	≥ 100
FRLDM	1.716	0.857	0.568	0.654	0.746	1.091
FRLDMC	1.307	0.900	0.582	0.814	0.494	0.508
FRDM	1.447	0.871	0.579	0.449	0.388	0.512
FRDMC	1.287	0.887	0.547	0.448	0.407	0.487

5. Conclusion

Based on a purely geometric interpretation of collective mass-parameters, the collective curvature scalar term has been introduced. For the geometry of the symmetric two-center shell model this term has been derived analytically. Interpreting this term as an additional potential term with the explicit form $V \sim A^{-5/3}$, we have investigated the influence of this term within the framework of two new macroscopic models: The finite range liquid drop model with curvature(FRLDMC) and the finite range droplet model with curvature (FRDMC).

Significant improvements have been found especially for light nuclei and for trans-fermium elements. Thus, the new models allow a more precise description of nuclear ground state properties.

Therefore, the collective curvature scalar, as a manifestation of interaction with curved collective space, plays an important role, e.g. for strong asymmetric fission, cluster-radioactivity or prediction of super heavy element properties.

6. References

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